Generating Random Variates

Chapter 8

8.1 Introduction

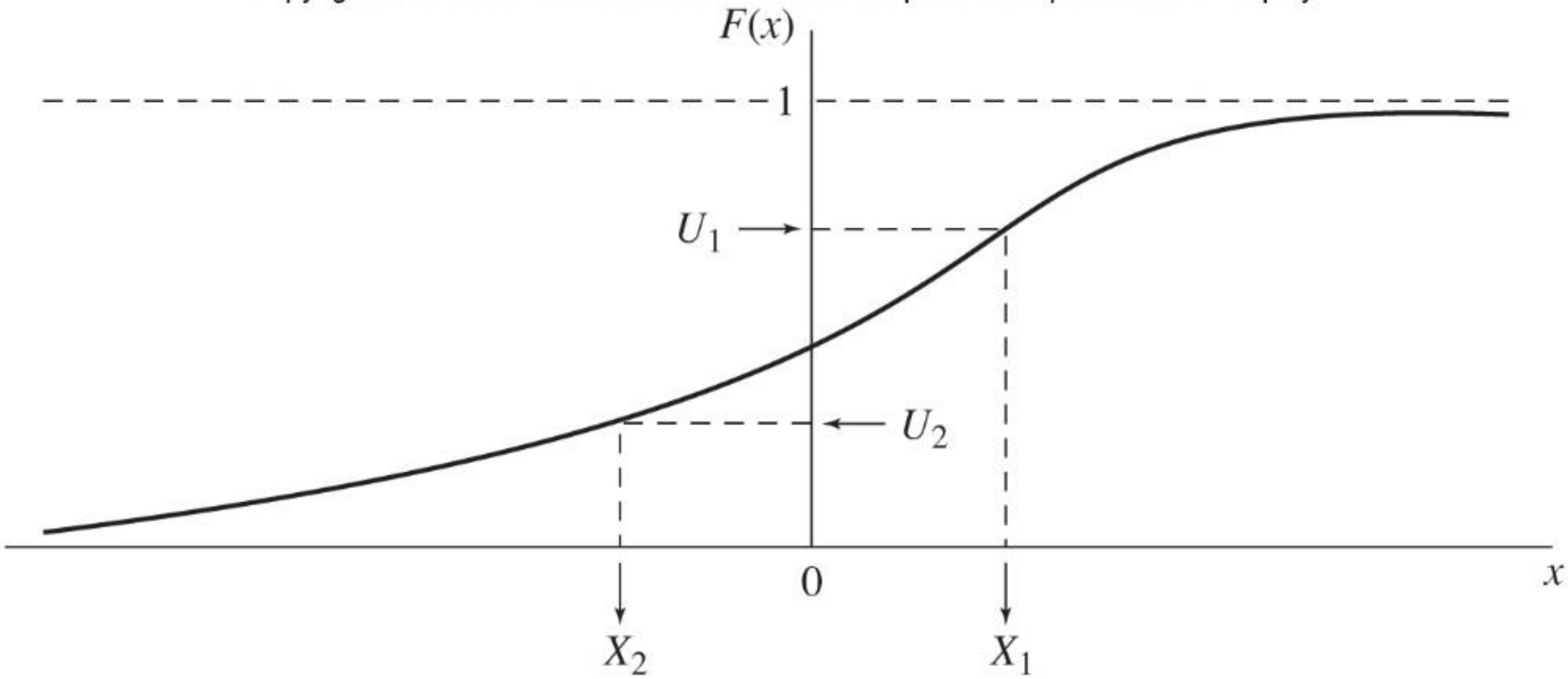
- Assume a distribution has already been specified
- What problem is addressed in this chapter?
  - How to generate random variates for use in a simulation model
- Basic ingredient
  - Source of IID $U(0,1)$ random variates
    - Statistically reliable $U(0,1)$ random number generator must be available

8.2 General Approaches to Generating Random Variates

- Inverse transform for continuous random var.
  - Suppose we wish to generate a continuous random variate, *X*
  - *X* has distribution function *F* that is continuous and strictly increasing when *F(x)* is between zero and one
  - Algorithm:

1. Generate $U \sim \mathrm{U}(0, 1)$.
2. Return $X = F^{-1}(U)$.

Inverse Transform for Continuous Random Variates



Inverse Transform for Continuous Random Variates Example

- Weibull(α,β)
- $F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ if x > 0
- Generate a Weibull(2,2) random variate

Inverse Transform for Truncated Continuous Random Variates

- $f^*(x) = \frac{f(x)}{F(b)-F(a)}$ a ≤ x ≤ b
- Algorithm
  - Generate U ~ U(0,1)
  - Let $V = F(a) + [F(b) - F(a)]U$
  - Return $X = F^{-1}(V)$

Inverse Transform for Continuous Random Variates Example

- Weibull(α,β)
- $F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ if x > 0
- Generate a Weibull(2,2) random variate truncated between [1,5]

Inverse Transform Method For Discrete Random Variates

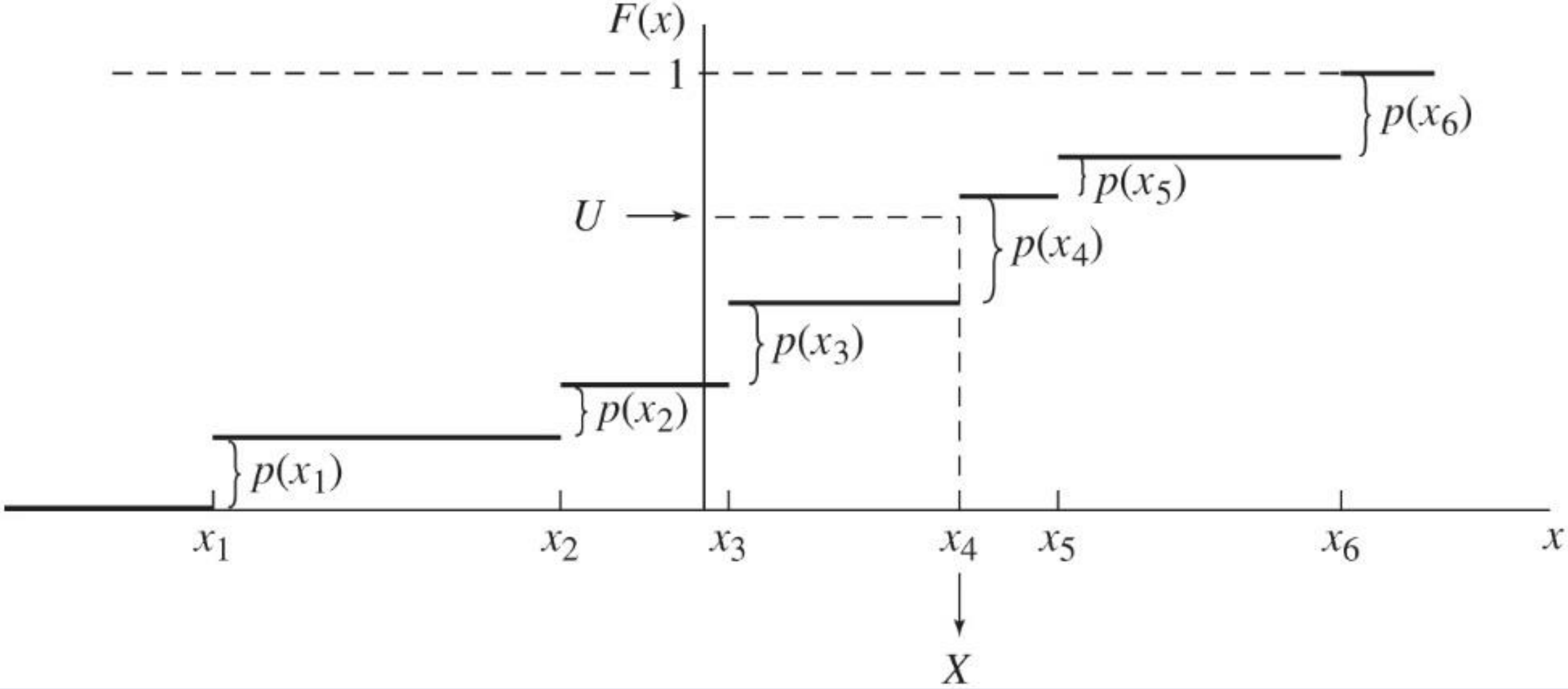
- Distribution function *F*(*x*) defined by:

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

$$p(x_i) = P(X = x_i)$$

- Algorithm:
    1. Generate $U \sim U(0,1)$
    2. Determine the smallest positive integer, I such that $U \leq F(x_I)$ and return $X = X_I$

Inverse Transform Method For Discrete Random Variates



HOWARD UNIVERSITY
Jiang Li, Ph.D., EECS

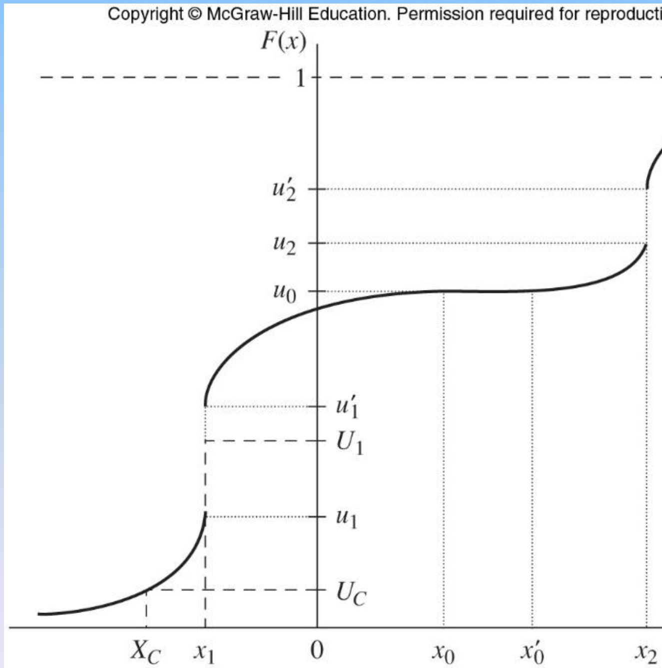
Inverse Transform for Continuous Random Variates Example

- $P(X = 1) = 1/6$
- $P(X = 2) = 1/3$
- $P(X = 3) = 1/3$
- $P(X = 4) = 1/6$

Generalized Inverse-Transform Method

- Works for a mixed distribution

$$X = \min\{x: F(x) \geq U\}$$



$P(X_1)=u_1'$
$P(X_2)=u_2$

HOWARD UNIVERSITY

Disadvantages

- In the continuous case
  - Might not be possible to write a formula for $F^{-1}(U)$
  - Numerical methods can be used
- May not be the fastest way

Composition Technique

- Applies when the distribution function *F* can be expressed as a convex combination of other distribution functions
- Assume that for all *x, F(x)* can be written as

$$F(x) = \sum_{j=1}^{\infty} p_j F_j(x) \qquad p_j \geq 0,\ \sum_{j=1}^{\infty} p_j = 1$$

- Composition algorithm
  1. Generate a positive random integer *J* such that:

  $$P(J = j) = p_j \qquad \text{for } j = 1, 2, \ldots$$

  2. Return *X* with distribution function F_J.

Composition Technique Example

- $f(x) = 0.5e^{x} I_{(-\infty,0)}(x) + 0.5e^{-x} I_{(0,\infty)}(x)$
  - $I_A(x) = 1 \ \text{ if } x \in A,\ I_A(x) = 0 \text{ otherwise}$
- Steps
  - Generate $U_1 \sim U(0,1)$
  - If $U_1 \leq 0.5$, generate $U_2 \sim U(0,1)$ return $X = \ln U_2$
  - If $U_1 > 0.5$, generate $U_2 \sim U(0,1)$ return $X = -\ln U_2$

Convolution Method

- For several important distributions, the desired random variable *X* can be expressed as the sum of other IID random variables
  – The other random variables *(Y)* can be generated more readily than *X*

$$X = Y_1 + Y_2 + \cdots + Y_m$$

- Convolution algorithm
  1. Generate $Y_1, Y_2,\ldots Y_m$ *IID* each with distribution function *G*
  2. Return X= $Y_1 + Y_2 + \ldots Y_m$

Convolution Method Example

- The m-Erlang r.v. X with mean β can be defined as the sum of m IID exponential r.v.s with common mean β/m
- To generate X
  - Generate $Y_1, Y_2, \ldots, Y_m$ as IID exponential with mean β/m
  - Return $X = Y_1 + Y_2 + \ldots + Y_m$