Output Data Analysis for a Single System

Chapter 9

9.1 Introduction

- Output data analysis is often not conducted appropriately
  - Treating output of a single simulation run as "true" system characteristics
- Appropriate statistical techniques must be used
  - Both in designing and analyzing system experiments

Random Nature of Simulation Output

- Output stochastic process from a single simulation run
  - $Y_1, Y_2, \ldots$
  - e.g. Y_i can be the delay of the i-th customer
- In general, Y_i's are not neither independent nor identically distributed
- First m output of j-th simulation run
  - $y_{j1}, y_{j2}, \ldots, y_{jm}$
- First m output of n simulation runs

$y_{11}, y_{12}, \ldots, y_{1m}$

$y_{21}, y_{22}, \ldots, y_{2m}$

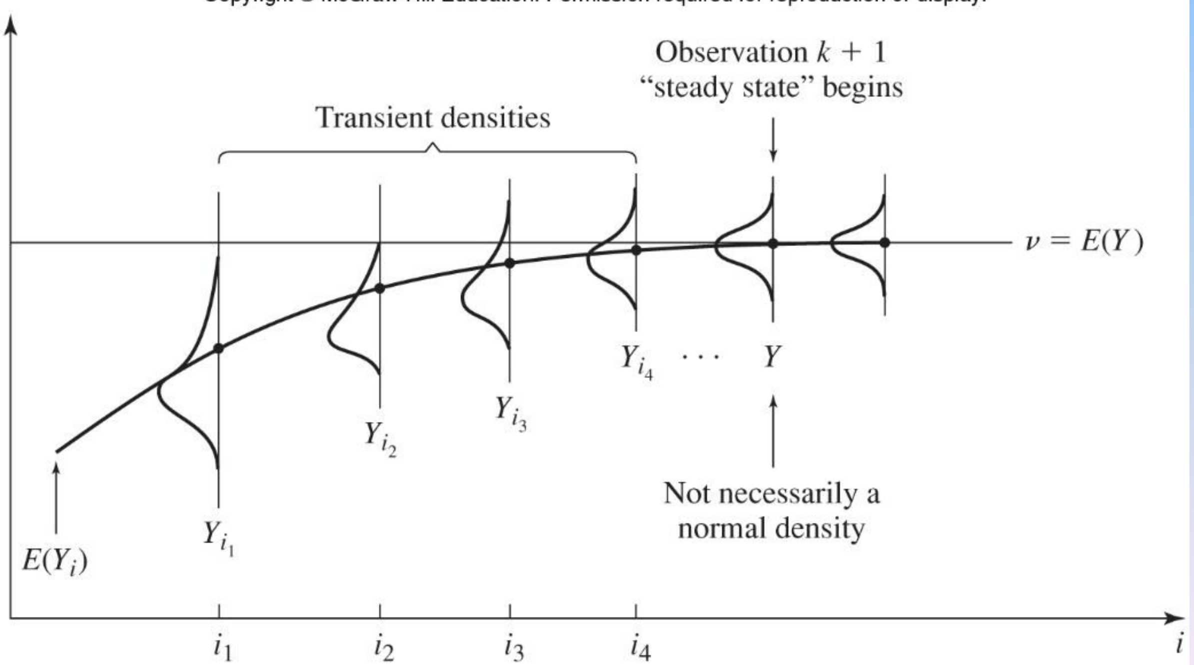
…

$y_{n1}, y_{n2}, \ldots, y_{nm}$

- $y_{1i}, \ldots, y_{ni}$ are IID observations of Y_i, can be used to infer about Y_i

9.2 Transient and Steady-State Behavior of a Stochastic Process

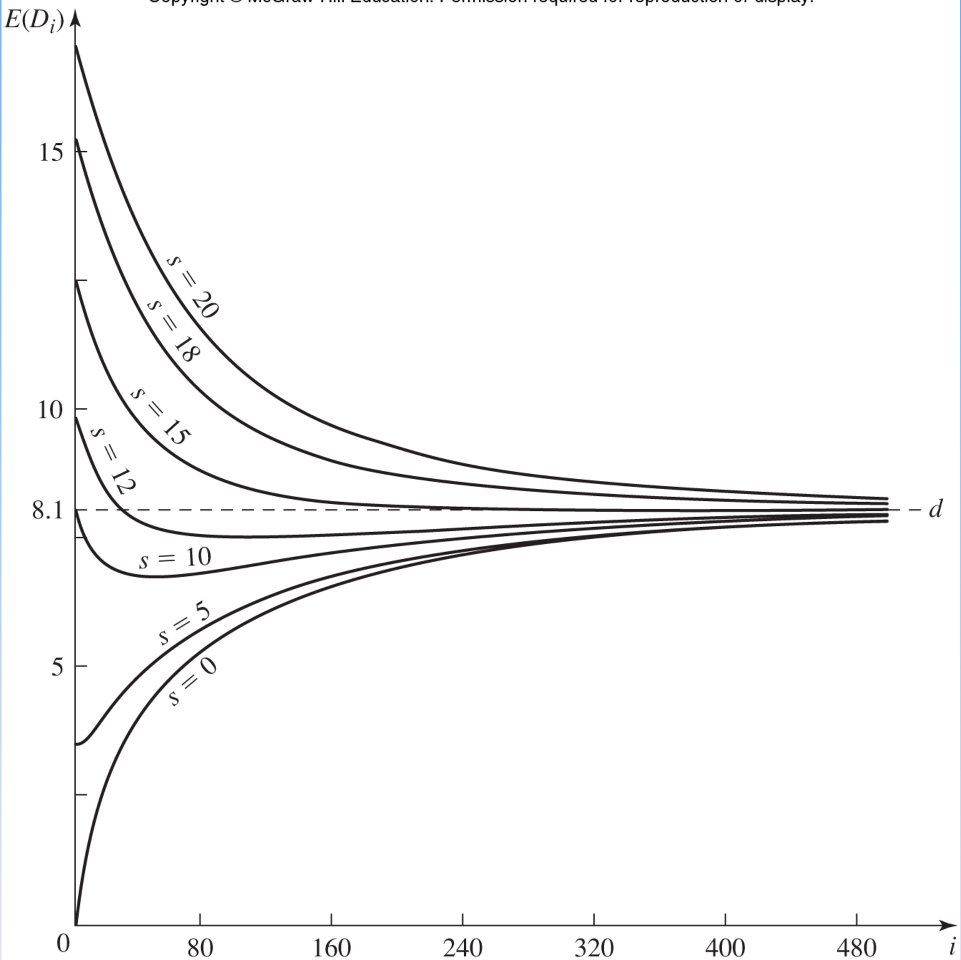
- Output stochastic process Y1,Y2,…
- Transient distribution
  - $F_i(y|I) = P(Y_i \le y|I)$ I: initial conditions
  - Usually different for each i and I
  - To make a histogram for $f_i(y|I)$, make n simulation runs and use the n observed values of Y_i

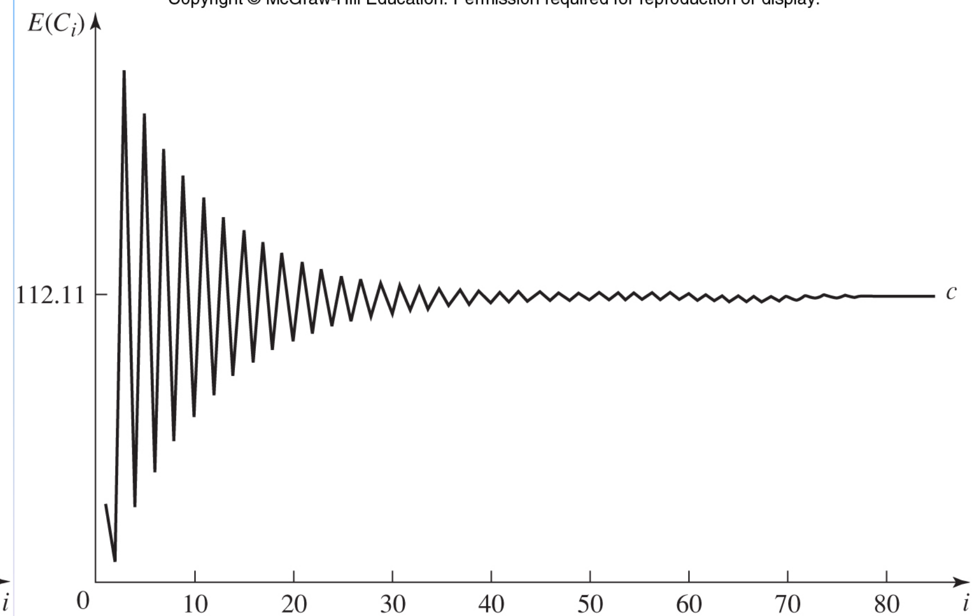


Steady-State Distribution

- $F_i(y|I) \rightarrow F(y)$ as $i \rightarrow \infty$
  - F(y): the steady state distribution of the output process
- In practice, find k such that $F_k(y|I), F_{k+1}(y|I), \ldots$ are approximately the same
  - Y_k, Y_{k+1}, … will approximately form a covariance-stationary stochastic process
  - Characteristics of $F_k(y|I), F_{k+1}(y|I)$, … will be similar
- F(y) does not depend on initial conditions
  - Converge rate does

Ex: Convergence to Steady State





9.3 Types of Simulations with Regard to Output Analysis

- Design and analysis depends on the type
- Simulation types
  - Terminating
    - A natural event specifies the length of each run
    - Initial conditions generally affect the performance measures, and thus should be representative
  - Nonterminating
    - There is no natural event to specify run length
    - We may be interested in the steady state mean or probability *P(Y<=y)* for some real number *y*

Terminating Events

- Terminating event is specified before running
  - May occur at a random time
- Depends on the objective of the simulation
- Examples
  - When the system is "cleaned out"
    - Last customer leaves after closing time (c.f. ending at closing time)
  - When no more useful information can be obtained
    - 30% of force is lost in military confrontation simulation
  - Points mandated by management
    - 10 years of inventory outlook for a company
    - 100 airplanes to produce

Nonterminating Simulations (1)

- Needed when we are not sure about the system's behavior
- Steady-state parameter
  - A characteristics of the steady-state distribution of some output stochastic process
  - Make sure new information from the real system is incorporated into the model as the latter may keep changing (and may not have steady states)
- E.g. hourly throughput of a production line

Nonterminating Simulations (2)

- A nonterminating simulation may not have a steady-state distribution
- Divide time into cycles
  - Y_i^C: random variable defined on the i-th cycle
    - E.g. average of production over a week
  - F^C: steady-state distribution of the process $Y_1^C, Y_2^C, \ldots$
  - Steady-state cycle parameter
    - A characteristic of Y^C ($Y^C \sim F^C$), e.g. mean

Steady-state Cycle Parameter Examples

- Manufacturing system with lunch break hour
  - No steady-state distribution for hourly throughput
  - Steady-state cycle parameter
    - Expected average hourly throughput in 8-hour cycles
- Call center with call arrival rate varying from day to day in a week
  - No steady-state distribution for the delay of i-th call
  - Steady-state cycle parameter
    - Expected average delay over a week

9.4 Statistical Analysis for Terminating Simulations

- Suppose we make *n* independent replications of a terminating simulation
  - Begin with the same initial conditions
  - Use different random numbers for each replication
  - Assume a single performance measure is of interest
    - E.g. the average delay of customers over a day (simulations terminate at the end of day),
    - E.g. number of tanks destroyed in a battle

Estimating Means

- X_i: a random variable defined on the i-th replication of simulation
- X_i's are IID
- Unbiased point estimate of μ: $\bar{X}(n) = \frac{\sum_{(i=1)}^{n} X_i}{n}$
- 100(1-α) percent confidence interval for μ

$$\bar{X}(n) \pm t_{n-1,1-\frac{\alpha}{2}} \sqrt{\frac{s^{2(n)}}{n}}$$

  – Fixed-sample-size procedure

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Example of Estimating Means

Replication	Number served	Finish time (hours)	Average delay in queue (minutes)	Average queue length	Proportion of customers delayed < 5 minutes
1	484	8.12	1.53	1.52	0.917
2	475	8.14	1.66	1.62	0.916
3	484	8.19	1.24	1.23	0.952
4	483	8.03	2.34	2.34	0.822
5	455	8.03	2.00	1.89	0.840
6	461	8.32	1.69	1.56	0.866
7	451	8.09	2.69	2.50	0.783
8	486	8.19	2.86	2.83	0.782
9	502	8.15	1.70	1.74	0.873
10	475	8.24	2.60	2.50	0.779

- $X_i = \frac{\sum_{j=1}^{N_i} D_j}{N_i}$ (N_i: number served)
- $\bar{X}(10) = 2.03, S^2(10) = 0.31$
- $\bar{X}(10) \pm t_{9,0.95}\sqrt{\frac{S^2(10)}{10}} = 2.03 \pm 0.32$

Robustness of the C.I. (1)

- How good is the coverage
- M/M/1 queue, ρ = 0.9, empty queue initially
  - Expected average delay of first 25 customers is 2.12
- 500 experiments
  - n replications of simulation per experiment (n=5,10,20,40)
  - Construct a 90% confidence interval of average delay for each experiment
  - $\hat{p}$: the proportion of the 500 C.I.'s containing 2.12
  - $\frac{t_{n-1,1-\frac{\alpha}{2}}\sqrt{\frac{S^2(n)}{n}}}{\bar{X}(n)}$: measurement of the precision of the confidence interval

Robustness of the C.I. (2)

- $\hat{p}$ is only an estimate of the true coverage
- Confidence interval of coverage

$$\hat{p} \pm Z_{0.95}\sqrt{\frac{\hat{p}(1-\hat{p})}{500}}$$

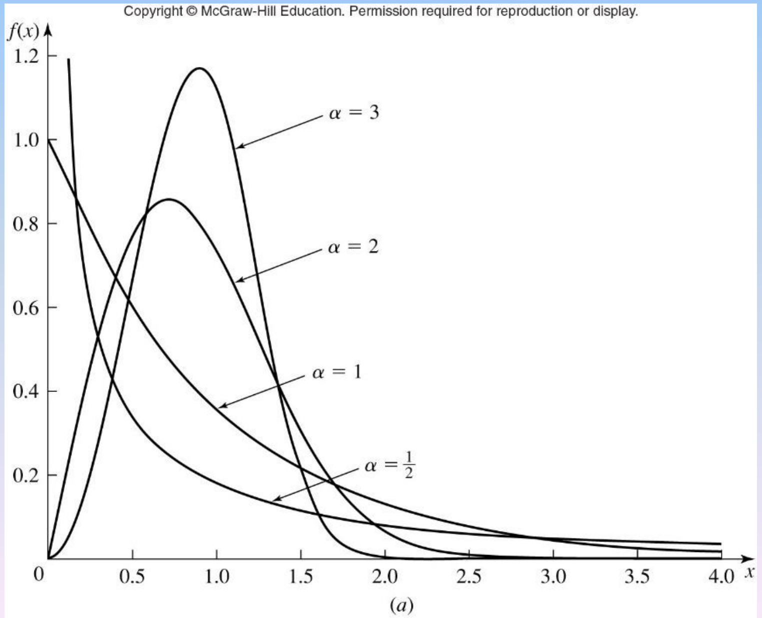
  – Should use only if $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$

n	Estimated coverage	Average of (confidence-interval half-length)/$\overline{X}(n)$
5	0.880 ± 0.024	0.67
10	0.864 ± 0.025	0.44
20	0.886 ± 0.023	0.30
40	0.914 ± 0.021	0.21

Four times of replications increases the precision by about 2

Robustness of the C.I. (3)

- Coverage may not be close to 1 - α
- Reliability model
  - Failure time $G = \min(G_1, \max(G_2, G_3))$
  - G_i's are independent r.v. of Weibull (0.5, 1)
  - n replications of simulation per experiment (n=5,10,20,40)
  - Construct a 90% confidence interval of failure time for each experiment



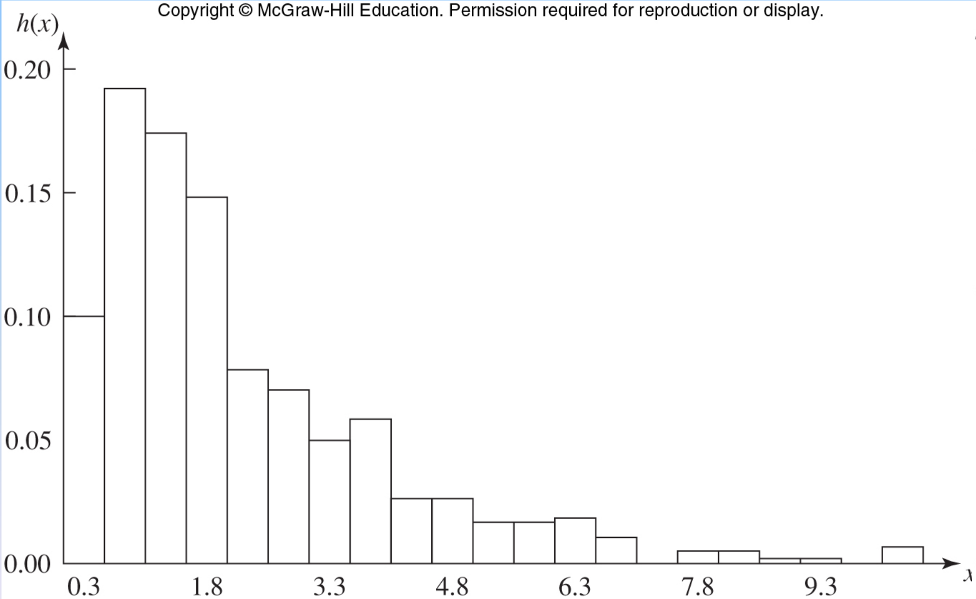
Robustness of the C.I. (4)

n	Estimated coverage	Average of (confidence-interval half-length)/$\overline{X}(n)$
5	0.708 ± 0.033	1.16
10	0.750 ± 0.032	0.82
20	0.800 ± 0.029	0.60
40	0.840 ± 0.027	0.44

- C.I. coverage depends on the distributions of X_i and replication number n
- Why does it work better for M/M/1?

Robustness of the C.I. (5)

- Histogram of average delay (based on 500 samples)
- Histogram of failure time (based on 500 samples)



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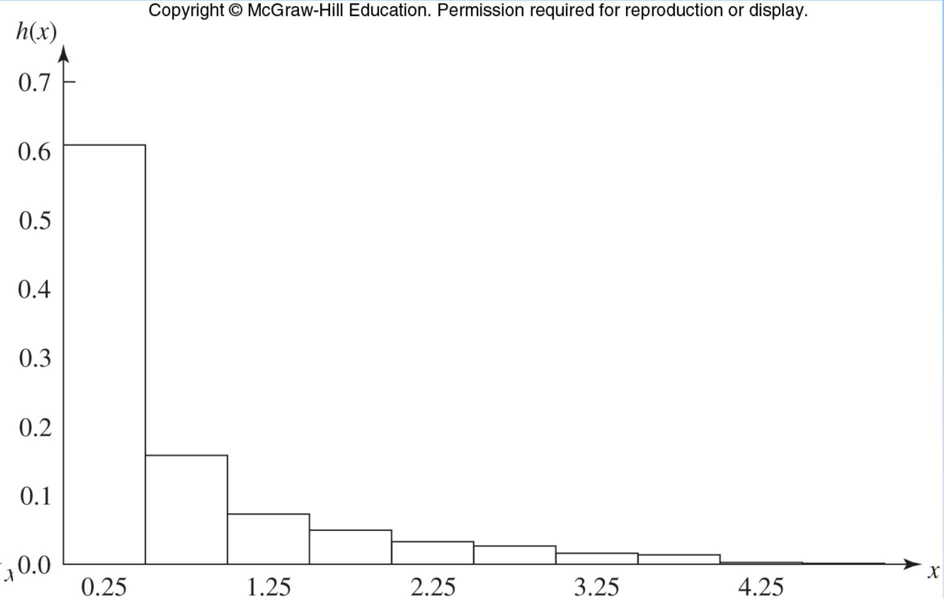


Obtaining a Specified Precision (1)

- Confidence interval precision depends on n
  - How much should be n for a particular precision?
- Measuring the error in $\bar{X}$
  - Absolute error of β

$$|\bar{X} - \mu| = \beta$$

  - Relative error of γ

$$\frac{|\bar{X} - \mu|}{|\mu|} = \gamma$$

Obtain Absolute Error of β

- Given a 100(1-α)% confidence interval
  - Let half length be h

$$1-\alpha \approx P(\bar{X}-h \leq \mu \leq \bar{X}+h) = P(|\bar{X}-\mu| \leq h)$$

- If h <= β, $P(|\bar{X}-\mu| \leq h) \leq P(|\bar{X}-\mu| \leq \beta)$
  - The absolute error is at most β w.p. 1 - α
- $\mathrm{h} = t_{n-1,1-\frac{\alpha}{2}}\sqrt{\frac{S^2(n)}{n}}$
  - Assume $S^2(n)$ does not change much as n changes (accuracy affected)
  - Find $n_a^*(\beta)$ as the smallest *i* s.t. $t_{i-1,1-\frac{\alpha}{2}}\sqrt{\frac{S^2(n)}{i}} \leq \beta$

$$n_a^*(\beta) = \min\left\{i \geq n : t_{i-1,1-\frac{\alpha}{2}}\sqrt{\frac{S^2(n)}{i}} \leq \beta\right\}$$

Obtaining Absolute Error Example

- Average customer delay in bank
  - $X_i = \frac{\sum_{j=1}^{N_i} D_j}{N_i}$ (N_i: number served)
  - $\bar{X}(10) = 2.03, S^2(10) = 0.31$
  - $\bar{X}(10) \pm t_{9,0.95}\sqrt{\frac{S^2(10)}{10}} = 2.03 \pm 0.32$
- Use absolute error of 0.25, 90% confidence interval

$$n_a^*(0.25) = \min\left\{ i \geq 10 : t_{i-1,0.95}\sqrt{\frac{0.31}{i}} \leq 0.25 \right\}$$

$$n_a^*(0.25) = 16$$

Obtain Relative Error of γ

- Given a 100(1-α)% confidence interval (half length: h)
$$1-\alpha \approx P(\bar{X}-h \le \mu \le \bar{X}+h) = P(|\bar{X}-\mu| \le h)$$
- If $\frac{h}{|\bar{X}|} \le \gamma'$, $P(|\bar{X}-\mu| \le \gamma' |\bar{X}|)$
$$\le P\left(|\bar{X}-\mu| \le \gamma'(|\bar{X}-\mu| + |\mu|)\right)$$
$$= P\left(\frac{|\bar{X}-\mu|}{|\mu|} \le \frac{\gamma'}{(1-\gamma')}\right)$$
- To get $1-\alpha \le P\left(\frac{\left|\bar{X}-\mu\right|}{|\mu|} \le \gamma\right)$, $\frac{h}{|\bar{X}|} \le \frac{\gamma}{1+\gamma}$
- Assume $S^2(n)$ and $\bar{X}(n)$ does not change much as n changes (accuracy affected)
$$n_r^*(\gamma) = \min\left\{ i \ge n: \frac{t_{i-1,1-\frac{\alpha}{2}}\sqrt{\frac{S^2(n)}{i}}}{|\bar{X}(n)|} \le \frac{\gamma}{1+\gamma} \right\}$$

Obtaining Relative Error Example

- Average customer delay in bank
  - $X_i = \frac{\sum_{j=1}^{N_i} D_j}{N_i}$ (N_i: number served)
  - $\bar{X}(10) = 2.03, S^2(10) = 0.31$
  - $\bar{X}(10) \pm t_{9,0.95}\sqrt{\frac{S^2(10)}{10}} = 2.03 \pm 0.32$
- Use relative error of 0.1, 90% confidence interval

$$n_r^*(0.1) = \min\left\{i \geq 10: \frac{t_{i-1,0.95}\sqrt{\frac{0.31}{i}}}{2.03} \leq \frac{0.1}{1+0.1}\right\}$$

$$n_r^*(0.1) = 27$$

Sequential Procedure

- New applications are added one at a time
- $\delta(n,\alpha) = t_{n-1,1-\frac{\alpha}{2}}\sqrt{\frac{S^2(n)}{n}}$
- γ: relative error (<= 0.15)
- Confidence level of 100(1 - α)%
- Steps
  1. n = n_0 (>= 10)
  2. Run n replications of the simulation
  3. Calculate $\bar{X}(n)$ and $\delta(n,\alpha)$
  4. If $\frac{\delta(n,\alpha)}{\bar{X}(n)} \leq \frac{\gamma}{1+\gamma}$, done; else n <- n + 1, go to step 2.

Using the procedure on the previous example yields n = 74

Recommended Use of the Procedures

- If the precision of the C.I. is not important, use the fixed-sample-size procedure
  - Willink C.I. may be needed if X_j's are highly non-normal and the number of replications is too small
- Otherwise, run n (>= 10) replications of the simulation, calculate $n_a^*(\beta)$ or $n_r^*(\gamma)$ and run more replications
  - If $\gamma > 0.15$ or small β, use several successive applications of the fixed-sample-size procedure

Choosing Initial Conditions

- Example:
  - Want to determine average delay of bank customers arriving between noon and 1 pm
  - Using initial condition of zero customers at noon will cause expected average delay to be biased low
  - One solution: begin simulation at 9am (bank opening time) with zero customers
    - Run for four simulated hours
  - Another approach for bank example
    - Collect data on the number of customers present in the bank at noon for several different days
      - Choose initial conditions randomly from the distribution

9.5 Statistical Analysis for Steady-State Parameters

- Problem of the initial transient
  - Also known as the startup problem
  - Output stochastic process: $Y_1, Y_2, \ldots$
  - Suppose $P(Y_i \leq y) = F_i(y) \to F(y) = P(Y \leq y)$ as $i \to \infty$
  - ϕ is a characteristic of Y (e.g. E(Y), a quantile of Y)
    - The characteristic based on $Y_1, Y_2, \ldots, Y_m$ is not representative of ϕ
    - E.g. $\bar{Y}(m)$ is a biased estimator of E(Y)
- Suggested solution: warming up the model or initial data deletion
  - Delete some observations from the beginning of a run

Warm Up The Model

- For example, use $\bar{Y}(m,l) = \frac{\sum_{i=l+1}^{m} Y_i}{m-l}$ to estimate E(Y)
- How to choose warmup period l
  - Pick l and m such that $E[\bar{Y}(m,l)] \approx E(Y)$
  - l and m can't be too small, or $E[\bar{Y}(m,l)] \neq E(Y)$
  - If l is larger than necessary, the variance of $\bar{Y}(m,l)$ will be unnecessarily large

Welch's Graphical Procedure

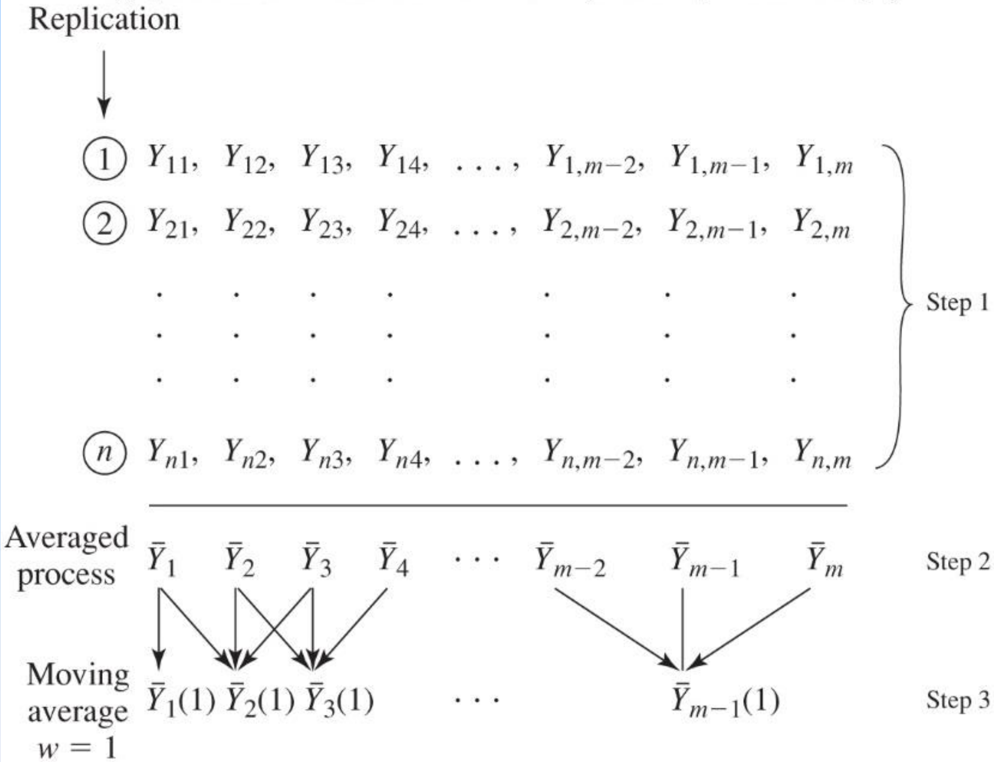
- n ≥ 5
- $\bar{Y}_i = \sum_{(j=1)}^{n} \frac{Y_{ji}}{n}$
- w: window
- If $i = 1, \dots, w$

$$\bar{Y}_i(w) = \frac{\sum_{s=-(i-1)}^{i-1} \bar{Y}_{i+s}}{2i-1}$$

- If $i = w+1, \dots, m-w$

$$\bar{Y}_i(w) = \frac{\sum_{s=-w}^{w} \bar{Y}_{i+s}}{2w+1}$$

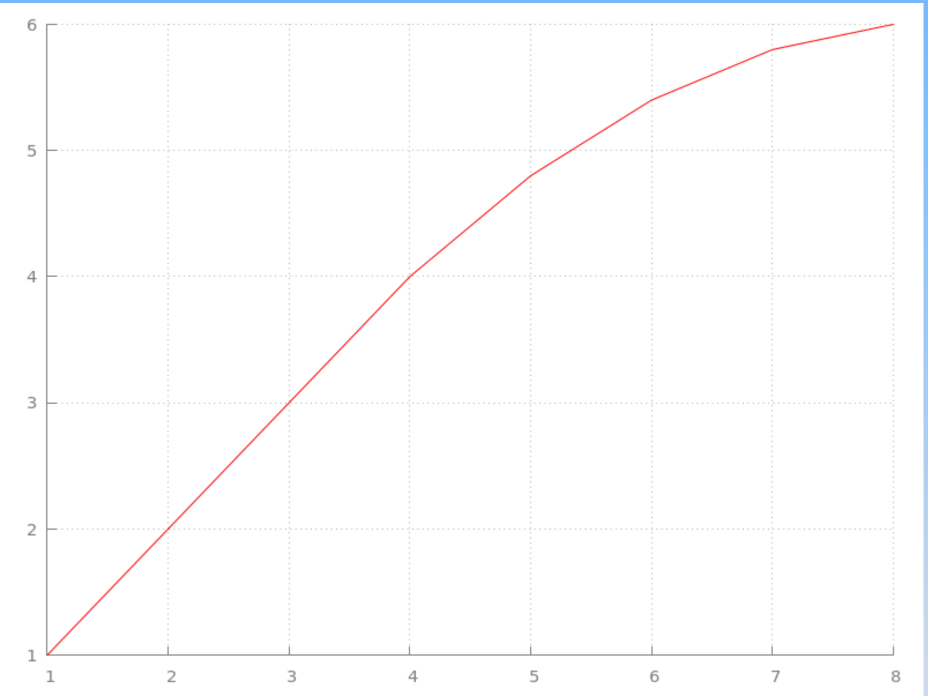
- Plot $\bar{Y}_i(w)$ for i = 1,2, …, m-w



Welch's Procedure Example 1

- m=10
- w=2
- $\bar{Y}_i = i$ for i = 1,2,..,5
- $\bar{Y}_i$ =6 for i = 6,7,...,10

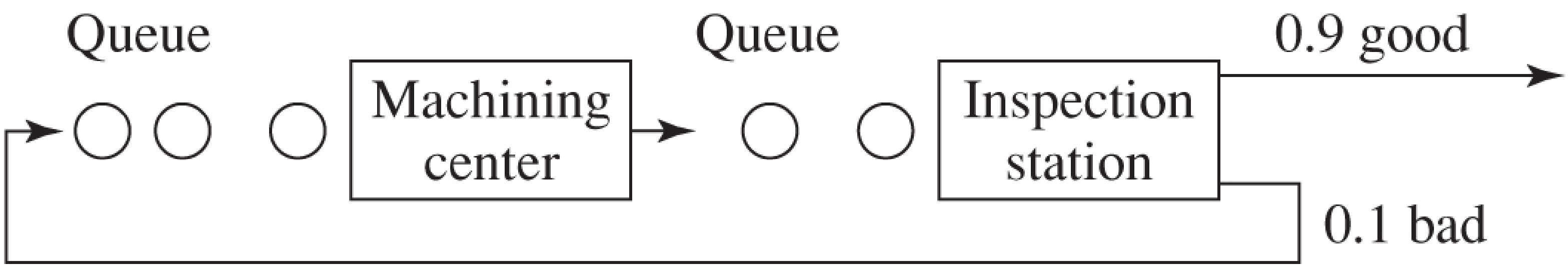
$\bar{Y}_1(2) = 1$	$\bar{Y}_2(2) = 2$
$\bar{Y}_3(2) = 3$	$\bar{Y}_4(2) = 4$
$\bar{Y}_5(2) = 4.8$	$\bar{Y}_6(2) = 5.4$
$\bar{Y}_7(2) = 5.8$	$\bar{Y}_8(2) = 6$

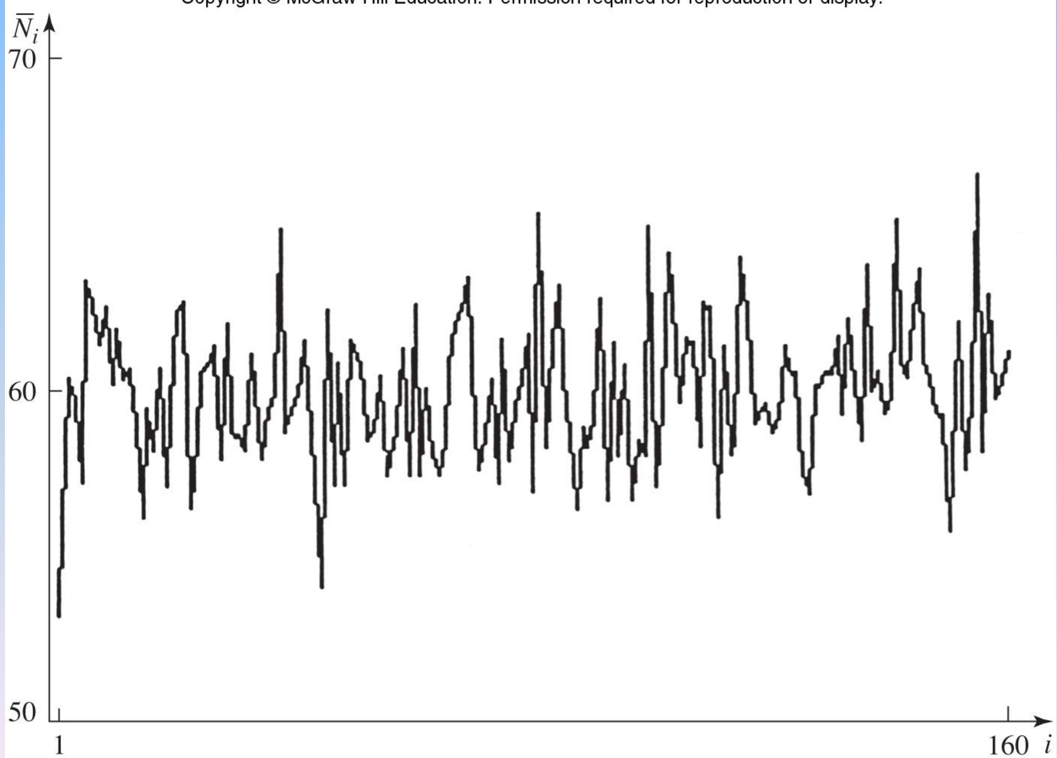


Recommendations for Welch's Procedure

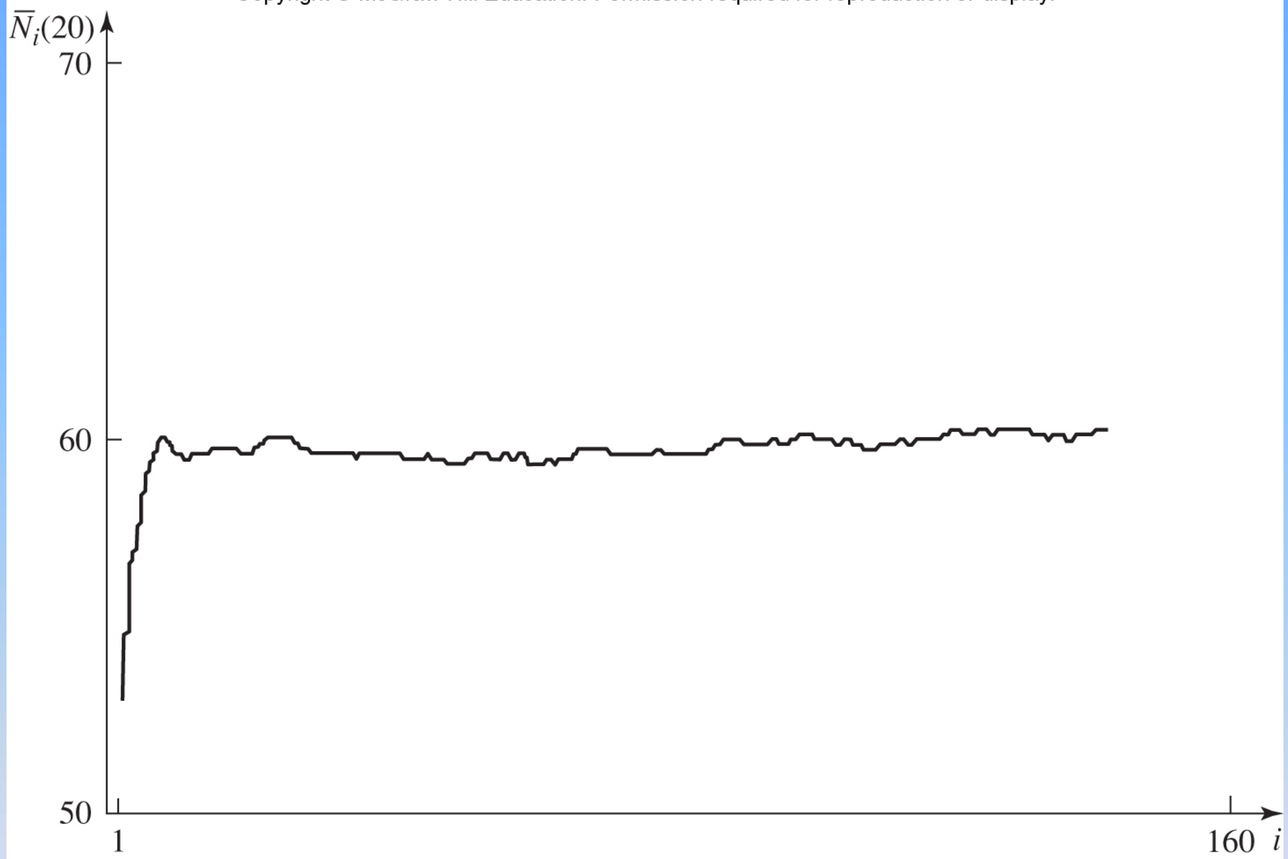
- Initially, n = 5 or 10, m should be much larger than anticipated l and large enough to allow for infrequent events
- Try different w for plotting, choose the smallest w of "reasonably smooth" plots
  - If no value of w looks good, $n \leftarrow n + (5 \text{ or } 10)$ and repeat

Welch's Procedure Example 2



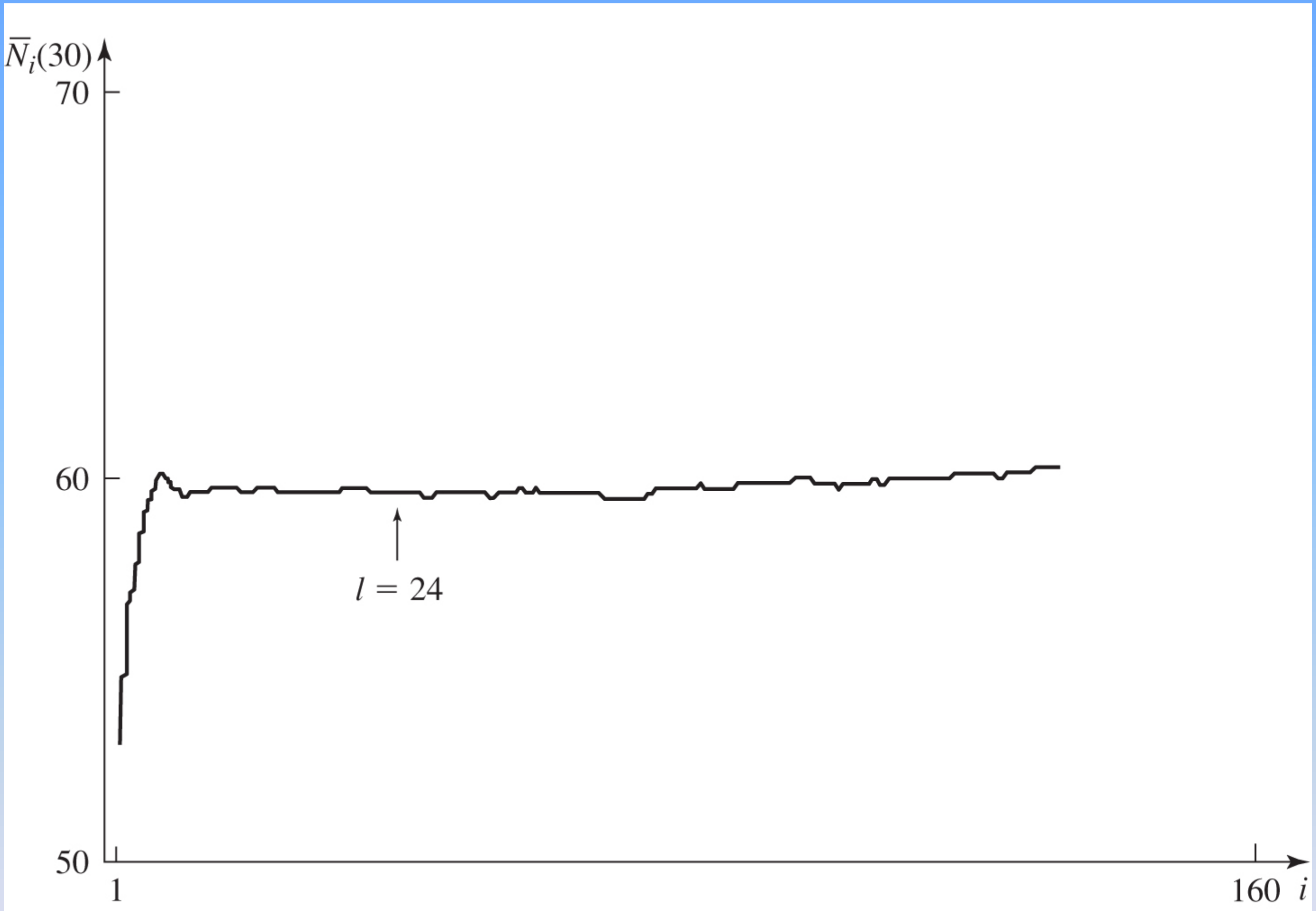


Welch's Procedure Example 2 (cont'd)



w = 20

Welch's Procedure Example 2 (cont'd)



w = 30

White's MSER Procedure

- Marginal Standard Error Rules
- $MSER(m,l) = \frac{\sum_{i=l+1}^{m}[Y_i - \bar{Y}(m,l)]^2}{(m-l)^2}$
- $l^* = \underset{l=0,1,\dots,m-1}{\arg\min}\ MSER(m,l)$
- MSER-k
  - Use batch average

    for j = 1,2,…,$\lfloor m/k \rfloor$

$$Z_j = \frac{\sum_{i=1}^{k} Y_{k(j-1)} + i}{k}$$

  - If l^* > $\lfloor m/k \rfloor/2$, increase m and redo.

Replication/Deletion Approach for Means

- Used to estimate the steady-state mean of a process
- Should give reasonably good statistical performance
- Easiest method to understand and implement
- Applies to all types of output parameters
- Can be used to estimate several different parameters

Replication/Deletion Approach for Means

- Similar to that for terminating simulations except that observations during warmup are not used
- Given n replications, for j-th replication
  - $X_j = \frac{\sum_{i=l+1}^{m} Y_{ji}}{m-l}$
  - $\bar{X}(n)$ is an approx. unbiased point estimate for E(Y)
  - C.I.

$$\bar{X}(n) \pm t_{n-1,1-\frac{\alpha}{2}} \sqrt{\frac{s^{2(n)}}{n}}$$

Replication/Deletion Approach for Means (cont'd)

- Use some replications as pilot runs for l, use another set for estimation
  - Can use the same set if m >> l
- To decrease C.I. half-length by k, make k^2 times as many replications