CSCI 410: Modeling and Simulation

Written Assignment 4 Solutions

1. If the following data are hypothesized to be drawn from an exponential distribution,
   a) **(5 pts)** Calculate the MLE of the parameter.
   b) **(10 pts)** Draw a Q-Q plot for the distribution with the estimated parameter and the data.
   5.07, 2.70, 0.29, 1.23, 4.00, 5.18, 2.63, 1.12, 1.43, 8.27
   7.51, 2.04, 3.05, 1.50, 3.14, 4.98, 1.91, 1.54, 8.07, 6.24
   5.82, 0.71, 9.53, 4.60, 5.76, 0.71, 9.50, 1.03, 0.70, 2.78

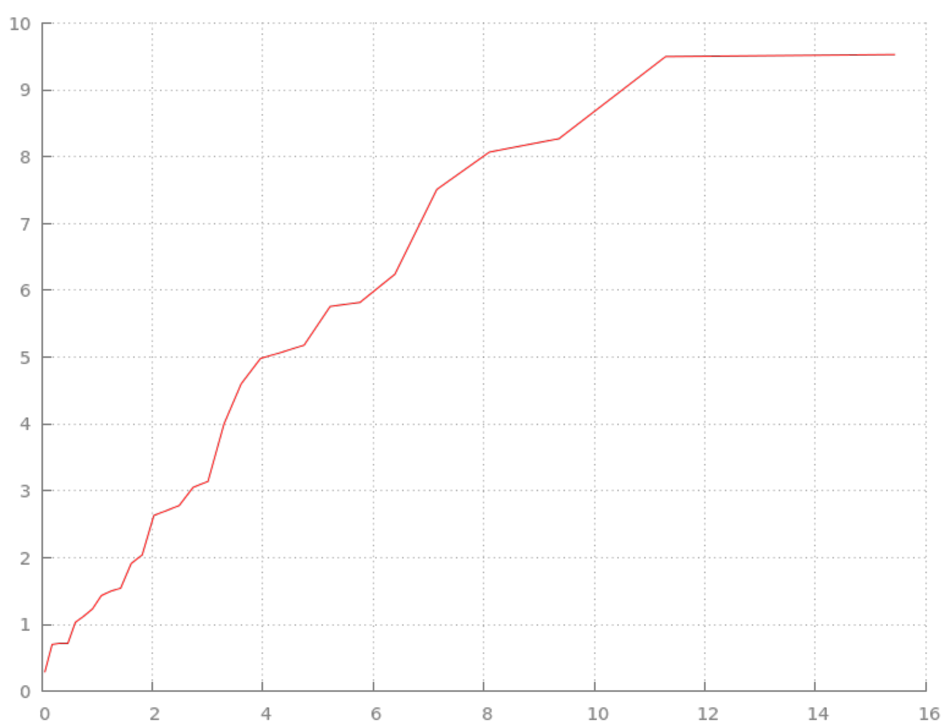
Answer:

a) For exponential distribution, the MLE of the parameter is the estimated mean. Therefore,

$\hat{\beta} = \bar{X}(30) \approx 3.768$

The density function is thus $f(x) = \begin{cases} \frac{1}{3.768} e^{-\frac{x}{3.768}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

b) (See the quantile data in the attached Excel sheet)



2. If the following data are hypothesized to be drawn from a Poisson distribution,
   a) **(5 pts)** Calculate the MLE of the parameter.

b) **(10 pts)** Draw a P-P plot for the distribution with the estimated parameter and the data.

3, 0, 7, 2, 6, 4, 3, 4, 3, 4, 5, 0
0, 3, 2, 1, 0, 3, 4, 4, 7, 2, 3, 6
2, 1, 2, 2, 1, 1, 3, 0, 3, 4, 3, 5

Answer:

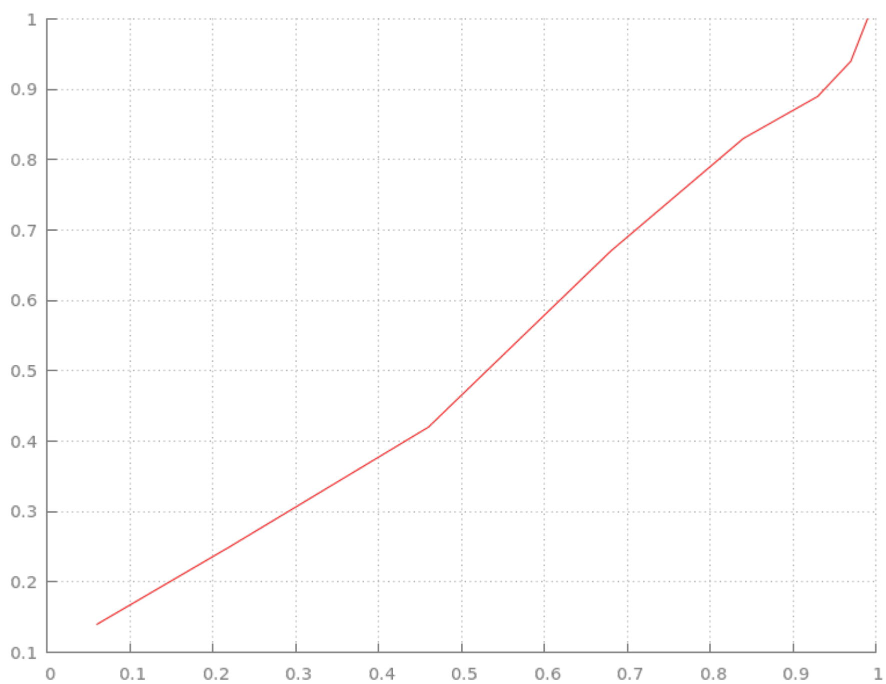
a) For Poisson distribution, the MLE of the parameter is also the estimated mean. Therefore,

$$\hat{\lambda} = \bar{X}(30) \approx 2.86$$

The mass function is thus $p(x) = \begin{cases} \frac{e^{-2.86} 2.86^x}{x!} & \text{if } x \in \{0,1,\dots\} \\ 0 & \text{otherwise} \end{cases}$

b) The empirical distribution is as follows.

P(X = 0) = 0.14
P(X = 1) = 0.11
P(X = 2) = 0.17
P(X = 3) = 0.25
P(X = 4) = 0.17
P(X = 5) = 0.06
P(X = 6) = 0.06
P(X = 7) = 0.06
(See the quantile data in the attached Excel sheet)



3. If the following data are hypothesized to be drawn from a gamma distribution,

a) **(10 pts)** Calculate the MLE of the parameters using Table 6.21 on the textbook.

b) **(Extra credit: 20 pts)** Perform the Chi-square test for the distribution and the data.
1.48, 0.24, 1.95, 0.20, 0.58, 1.38, 0.04, 0.55, 4.56, 3.96
1.20, 1.79, 0.23, 0.08, 0.56, 2.11, 1.38, 0.09, 0.12, 0.21
0.68, 0.05, 3.24, 2.52, 0.01, 0.00, 2.06, 0.85, 0.76, 3.55

Answer:

a) To calculate the MLE of the parameters, according to the formula in Figure 6.7, we need to calculate ln (X_i). Since ln(0) is not defined, we discard the sample values being 0.

$T = \left[\ln \bar{X}(n) - \sum_{j=1}^{n} \frac{\ln X_i}{n}\right]^{-1} = 1.27$ (See the attached Excel sheet for calculation details)

According to Table 6.21, $\hat{\alpha} \approx 0.775$.

$$\hat{\beta} = \frac{\bar{X}(29)}{\hat{\alpha}} = \frac{1.26}{0.775} \approx 1.626$$

The density function is $f(x) = \begin{cases} \frac{0.686x^{-0.225}e^{-\frac{x}{1.626}}}{\Gamma(0.775)} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

b) $\Gamma(0.775) \approx 1.19353$

Suppose we use five intervals. $\hat{F}(a_j) = \frac{j}{5}$ for j = 0,1,…,5. Using numerical methods, we have

$a_0 = 0$

$a_1 = 0.197$ ($\int_0^{0.197} f(x)dx \approx 0.2$)

$a_2 = 0.541$ ($\int_0^{0.541} f(x)dx \approx 0.4$)

$a_3 = 1.079$ ($\int_0^{1.079} f(x)dx \approx 0.6$)

$a_4 = 2.064$ ($\int_0^{2.064} f(x)dx \approx 0.8$)

$a_5 = \infty$

From the calculation shown in the attached Excel sheet, $\chi^2 = 1.00$

$\chi^2_{4,0.90} = 7.779$

$\chi^2_{4,0.95} = 9.488$

Since they are both larger than χ^2, we fail to reject the null hypotheses at both the 90% and 95% level.

4. **(15 pts)** Fit the following data to a shifted log-normal distribution by calculating the MLE of the parameters.
159.5, 24.8, 270.7, 0.3, 23.0, 287.3, 90.2, 708.3, 3387.0, 6037.5, 909.1, 12.6
157.1, 141.8, 1.2, 329.2, 125.0, 22.4, 12.8, 1.1, 226.7, 3.6, 710.6, 2.1
363.8, 1.3, 18.9, 145.6, 54.6, 92.1, 4119.9, 490.5, 151.0, 359.4, 8.5, 12385.6

Answer:

The minimum of the values is 0.3, the second minimum is 1.1, and the maximum is 12385.6.

$$\tilde{\gamma} = \frac{X_{(1)}X_{(30)} - X_{(2)}^2}{X_{(1)} + X_{(30)} - 2X_{(2)}} = \frac{0.3 \times 12385.6 - 1.1^2}{0.3 + 12385.6 - 2 \times 1.1} \approx 0.30$$

The adjusted data $X_i - \tilde{\gamma}$ is thus

159.2, 24.5, 270.4, 0, 22.7, 287, 89.9, 708, 3386.7, 6037.2, 908.8, 12.3
156.8, 141.5, 0.9, 328.9, 124.7, 22.1, 12.5, 0.8, 226.4, 3.3, 710.3, 1.8
363.5, 1, 18.6, 145.3, 54.3, 91.8, 4119.6, 490.2, 150.7, 359.1, 8.2, 12385.3

(Value 0 is discarded. See the attached Excel sheet for calculation details.)

$\hat{\mu} = \frac{\sum_{i=1}^{29} \ln X_i}{29} \approx 4.54$

$\hat{\sigma} = \left[\frac{\sum_{i=1}^{29} (\ln X_i - \hat{\mu})^2}{29}\right]^{\frac{1}{2}} \approx 2.44$

The density function is thus

$$f(x) = \begin{cases} \frac{0.164}{x} e^{-\frac{(\ln(x-0.3)-4.54)^2}{11.91}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

5. **(10 pts)** The distance of marathon is 42.195 kilometers. The world record is about 2 hours and 3 minutes or 123 minutes. It is known that people's normal walking speed ranges from 4.5 to 6.5 kilometers per hour. Suppose in the worst case, a marathon athlete would finish the race by walking. Also suppose the most likely time to finish a marathon race is four and a half hours. Use the triangle-distribution approach and beta-distribution approach respectively to specify the distribution of marathon running time in minutes. Round the numbers if necessary. Provide the density functions.

Answer:

Assume a person walks to finish marathon, the longest time is thus

42.195 / 4.5 × 60 ≈ 562.6 minutes

The triangle-distribution approach:

a = 123, b = 562.6, m = 4.5 × 60 = 260

Density function:

$$f(x) = \begin{cases} \frac{2(x-123)}{(562.6-123)(260-123)} = 0.000033x - 0.004085 & \text{if } 123 \le x \le 260 \\ \frac{2(562.6-x)}{(562.6-123)(260-123)} = 0.018683 - 0.000033x & \text{if } 260 < x \le 562.6 \\ 0 & \text{otherwise} \end{cases}$$

The beta-distribution approach:

Suppose we use the fastest walking time to get the average time to finish marathon.

μ = 42.195 / 6.5 × 60 ≈ 389.5 minutes

$$\tilde{\alpha}_1 = \frac{(\mu - a)(2m - a - b)}{(m - \mu)(b - a)} = \frac{(389.5 - 123)\times(2\times 260 - 123 - 562.6)}{(260 - 389.5)(562.6 - 123)} \approx 0.775$$

$$\tilde{\alpha}_2 = \frac{(b - \mu)\tilde{\alpha}_1}{\mu - a} = \frac{(562.6 - 389.5)\times 0.775}{389.5 - 123} \approx 0.503$$

Density function:

$$f(x) = \begin{cases} \dfrac{x^{-0.225}(1 - x)^{-0.497}}{B(0.775{,}0.503)} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$